## Solution to Homework Assignment No. 1

**1.** (a) (i) Perform elimination as follows:

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right] \text{(subtract } 1/2 \times \text{row 1)}$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right] \text{(subtract } 2/3 \times \text{row 2)}$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{array}\right] \text{(subtract } 3/4 \times \text{row 3)}$$

This system is equivalent to

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}.$$

Then we can solve the equations by back substitution as

$$\begin{cases} 2x + y = 0 \\ \frac{3}{2}y + z = 0 \\ \frac{4}{3}z + t = 0 \\ \frac{5}{4}t = 5 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2}y \\ y = -\frac{2}{3}z \\ z = -\frac{3}{4}t \\ t = 4 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = 2 \\ z = -3 \\ t = 4. \end{cases}$$

The pivots are 2, 3/2, 4/3, and 5/4, and the solution is $(x, y, z, t) = (-1, 2, -3, 4)$.

(ii) Perform elimination as follows:

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right] \text{(subtract } -1/2 \times \text{row 1)}$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right] \text{(subtract } -2/3 \times \text{row 2)}$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{array}\right] \text{(subtract } -3/4 \times \text{row 3)}$$

This system is equivalent to

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}.$$

Then we can solve the equations by back substitution as

$$\begin{cases} 2x - y = 0 \\ \frac{3}{2}y - z = 0 \\ \frac{4}{3}z - t = 0 \\ \frac{5}{4}t = 5 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}y \\ y = \frac{2}{3}z \\ z = \frac{3}{4}t \\ t = 4 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \\ z = 3 \\ t = 4. \end{cases}$$

The pivots are 2, 3/2, 4/3, and 5/4, and the solution is $(x, y, z, t) = (1, 2, 3, 4)$.

(b) Do elimination once more, and we can obtain the fifth pivot equal to $2 - (5/4)^{-1} = 6/5$. Observe the pivots, 2, 3/2, 4/3, 5/4, 6/5, $\cdots$, and we can guess that the $n$th pivot is equal to $(n+1)/n$.

**Claim:** The $n$th pivot is $(n+1)/n$.
**Proof:** When $n = 1$, the 1st pivot is 2/1.
Assume when $n = k-1$, the $k$th pivot is $k/(k-1)$.
By observing the procedure of elimination, we can know that the $k$th pivot is generated in the following way:

$$\text{the } k\text{th pivot} = 2 - \frac{1}{\text{the } (k-1)\text{th pivot}} = 2 - \frac{k-1}{k} = \frac{k+1}{k}.$$

By induction, we conclude that the $n$th pivot is $(n+1)/n$. ■

**2.** Pascal's triangle is defined as

1
1 1
1 2 1
1 3 3 1
...

The $i$th row of Pascal's triangle has $i$ components denoted as $[P_{i,1}, P_{i,2}, ..., P_{i,i}]$ which are derived by

$$[P_{i,1}, P_{i,2}, ..., P_{i,i}] = [P_{i-1,1}, P_{i-1,2}, ..., P_{i-1,i-1}, 0] + [0, P_{i-1,1}, P_{i-1,2}, ..., P_{i-1,i-1}].$$

For example, the components of the 4th row are given by

$$[1, 3, 3, 1] = [1, 2, 1, 0] + [0, 1, 2, 1].$$

To reduce the Pasical matrix to a smaller one as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

we can subtract the $(i-1)$th row from the $i$th row of the original matrix for $i = 2, 3, 4$. This process can be expressed by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Therefore, we have

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

In a similar way, we can reduce the Pascal matrix all the way to an identity matrix as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, we can have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The desired matrix is then given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

**3.** (a) Using the Gauss-Jordan method, we can have

$$\begin{aligned}
\left[\ \boldsymbol{A}\ \middle|\ \boldsymbol{I}\ \right] &= \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & \frac{3}{2} & 0 & -\frac{3}{8} & \frac{9}{8} & -\frac{3}{8} \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & 0 & -\frac{3}{8} & \frac{9}{8} & -\frac{3}{8} \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} \end{array}\right] = \left[\ \boldsymbol{I}\ \middle|\ \boldsymbol{A^{-1}}\ \right].
\end{aligned}$$

The inverse is hence

$$\boldsymbol{A^{-1}} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & -3 \end{bmatrix}.$$

(b) Using the Gauss-Jordan method, we can have

$$\begin{aligned}
\left[\ \boldsymbol{B}\ \middle|\ \boldsymbol{I}\ \right] &= \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 1 \end{array}\right] \\
&\Longrightarrow \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right].
\end{aligned}$$

Since we cannot obtain three nonzero pivots, $\boldsymbol{B}^{-1}$ does not exist.

**4.** (a) Using the Gauss-Jordan method, we can have

$$\left[\, \boldsymbol{A} \mid \boldsymbol{I} \,\right] = \left[\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \color{red}{1} & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & \color{red}{1} & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & \color{red}{1} & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\, \boldsymbol{I} \mid \boldsymbol{A}^{-1} \,\right].$$

We can then obtain

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Extend $\boldsymbol{A}$ to a $5 \times 5$ "alternating matrix" as

$$\boldsymbol{A}_{5\times 5} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the result of $(a)$, we guess

$$\boldsymbol{A}_{5\times 5}^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

and

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

we have confirmed that the inverse of the matrix is indeed

$$\boldsymbol{A}_{5\times 5}^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

**5.** (a) Performing elimination, we can have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \overset{\boldsymbol{E}_{21}}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{U}.$$

This procedure can be viewed as

$$\boldsymbol{E}_{32}\boldsymbol{E}_{21}\boldsymbol{A} = \boldsymbol{U}$$

where

$$\boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Then we have

$$\boldsymbol{A} = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{32}^{-1}\boldsymbol{U} = \boldsymbol{L}\boldsymbol{U}$$

where

$$\boldsymbol{L} = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

We find that $\boldsymbol{U} = \boldsymbol{L}^T = \boldsymbol{D}\boldsymbol{L}^T$ where $\boldsymbol{D} = \boldsymbol{I}$. We can therefore factor $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ and $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$ as

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Performing elimination, we can have

$$\boldsymbol{A}=\begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} \overset{\boldsymbol{E}_{21}}{\Longrightarrow} \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & b & b+c \end{bmatrix} \overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix} = \boldsymbol{U}.$$

Since $\boldsymbol{E}_{21}$ and $\boldsymbol{E}_{32}$ are the same as those in (a), we know that $\boldsymbol{A}$ has the same $\boldsymbol{L}$, too. The factorization $\boldsymbol{A}=\boldsymbol{L}\boldsymbol{U}$ is hence

$$\begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix}.$$

We can further factor $\boldsymbol{U}$ as

$$\boldsymbol{U}=\begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{D}\boldsymbol{L}^T.$$

The factorization $\boldsymbol{A}=\boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$ is thus given by

$$\begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

**6.** (a) Given $\boldsymbol{A}=\boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1$ and $\boldsymbol{A}=\boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2$, we can have

$$\begin{aligned} & \boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2 = \boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1 \\ \Longrightarrow\ & \boldsymbol{L}_1^{-1}(\boldsymbol{L}_2\boldsymbol{D}_2\boldsymbol{U}_2)\boldsymbol{U}_2^{-1} = \boldsymbol{L}_1^{-1}(\boldsymbol{L}_1\boldsymbol{D}_1\boldsymbol{U}_1)\boldsymbol{U}_2^{-1} \\ \Longrightarrow\ & \boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2 = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}. \end{aligned}$$

In order to explain why one side is lower triangular and the other side is upper triangular, we need to prove two claims first.

**Claim 1:** The inverse of a lower (upper) triangular matrix with unit diagonal is also lower (upper) triangular with unit diagonal.

**Proof:** (Lower triangular case)

Suppose $\boldsymbol{L}$ is an $n \times n$ lower triangular matrix with unit diagonal and $\boldsymbol{L}^{-1}$ exists. We can use Gauss-Jordan method to find $\boldsymbol{L}^{-1}$. We only need to do the Gaussian part. It means that the required operations are only to subtract the $i$th row from the $j$th row for $i<j$. Therefore, we can have

$$\begin{aligned} \left[\ \boldsymbol{L} \mid \boldsymbol{I}\ \right] &= \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ l_{2,1} & 1 & \ddots & \vdots & 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 & 0 & 1 & 0 \\ l_{n,1} & \cdots & l_{n,n-1} & 1 & 0 & 0 & 0 & 1 \end{array}\right] \\ &\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & l'_{2,1} & 1 & \ddots & \vdots \\ 0 & 0 & 1 & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & l'_{n,1} & \cdots & l'_{n,n-1} & 1 \end{array}\right] = \left[\ \boldsymbol{I} \mid \boldsymbol{L}^{-1}\ \right]. \end{aligned}$$

It is clear that $\boldsymbol{L}^{-1}$ is lower triangular with unit diagonal. The upper triangular case can be proved similarly. ■

**Claim 2:** The product of two lower (upper) triangular matrices with unit diagonal is also lower (upper) triangular with unit diagonal.

**Proof:** (Lower triangular case)

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are two $n \times n$ lower triangular matrices with unit diagonal. We have $A_{i,j} = 0$ if $i < j$ and $A_{i,j} = 1$ if $i = j$, and $B_{i,j} = 0$ if $i < j$ and $B_{i,j} = 1$ if $i = j$. For $1 \leq i < j \leq n$, we have

$$\begin{aligned}(AB)_{i,j} &= \sum_{k=1}^{n} A_{i,k}B_{k,j} \\ &= \sum_{k=1}^{j-1} A_{i,k}B_{k,j} + \sum_{k=j}^{n} A_{i,k}B_{k,j} \\ &= 0 + 0 \ (B_{i,k} = 0 \text{ when } k < j, \text{ and } A_{i,k} = 0 \text{ when } i < j \leq k.) \\ &= 0.\end{aligned}$$

Therefore, $\boldsymbol{AB}$ is lower triangular. For $1 \leq i = j \leq n$, we have

$$\begin{aligned}(AB)_{i,i} &= \sum_{k=1}^{n} A_{i,k}B_{k,i} \\ &= \sum_{k=1}^{i-1} A_{i,k}B_{k,i} + A_{i,i}B_{i,i} + \sum_{k=i+1}^{n} A_{i,k}B_{k,i} \\ &= 0 + 1 \cdot 1 + 0 \ (B_{i,k} = 0 \text{ when } k < i, \ A_{i,i} = B_{i,i} = 1, \text{ and } A_{i,k} = 0 \text{ when } i < k) \\ &= 1.\end{aligned}$$

Therefore, $\boldsymbol{AB}$ has unit diagonal. We can conclude that $\boldsymbol{AB}$ is also lower triangular with unit diagonal. The upper triangular case can be proved similarly. ■

Let $\boldsymbol{A} = \begin{bmatrix} \underline{\mathbf{a}}_1 \\ \underline{\mathbf{a}}_2 \\ \vdots \\ \underline{\mathbf{a}}_n \end{bmatrix}$, where $\underline{\mathbf{a}}_i = [a_{i,1} \ a_{i,2} \ \cdots \ a_{i,n}]$, and $\boldsymbol{D}$ be a diagonal matrix with diagonal elements $d_1$, $d_2$,..., $d_n$. We can have

$$\boldsymbol{AD} = \begin{bmatrix} \underline{\mathbf{a}}_1 \\ \underline{\mathbf{a}}_2 \\ \vdots \\ \underline{\mathbf{a}}_n \end{bmatrix} \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_n \end{bmatrix} = \begin{bmatrix} d_1\underline{\mathbf{a}}_1 \\ d_2\underline{\mathbf{a}}_2 \\ \vdots \\ d_n\underline{\mathbf{a}}_n \end{bmatrix}$$

and

$$\boldsymbol{DA} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_n \end{bmatrix} \begin{bmatrix} \underline{\mathbf{a}}_1 \\ \underline{\mathbf{a}}_2 \\ \vdots \\ \underline{\mathbf{a}}_n \end{bmatrix} = \begin{bmatrix} d_1\underline{\mathbf{a}}_1 \\ d_2\underline{\mathbf{a}}_2 \\ \vdots \\ d_n\underline{\mathbf{a}}_n \end{bmatrix}.$$

Therefore, a lower (upper) triangular matrix multiplied by a diagonal matrix is still a lower (upper) triangular matrix. Come back to $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2 = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$. By Claim 1, $\boldsymbol{L}_1^{-1}$ is lower triangular with unit diagonal. By Claim 2, $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2$ is lower triangular with unit diagonal. Therefore, $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2$ is lower triangular. Similarly, $\boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ is upper triangular.

(b) Let $\boldsymbol{M} = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2 = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$. Then $\boldsymbol{M}$ is both lower and upper triangular, which implies that $\boldsymbol{M}$ is a diagonal matrix.

  (i) Since $\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ is with unit diagonal, $\boldsymbol{M} = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1}$ has the same diagonal as $\boldsymbol{D}_1$. It implies that $\boldsymbol{M} = \boldsymbol{D}_1$. Similarly, we can have $\boldsymbol{M} = \boldsymbol{D}_2$. Therefore, $\boldsymbol{D}_1 = \boldsymbol{D}_2$.

  (ii) For $\boldsymbol{M} = \boldsymbol{L}_1^{-1}\boldsymbol{L}_2\boldsymbol{D}_2 = \boldsymbol{D}_2$, we have $\boldsymbol{L}_1^{-1}\boldsymbol{L}_2 = \boldsymbol{I}$. Since the inverse matrix is unique, we have $\boldsymbol{L}_2 = (\boldsymbol{L}_1^{-1})^{-1} = \boldsymbol{L}_1$.

  (iii) Similarly, for $\boldsymbol{M} = \boldsymbol{D}_1\boldsymbol{U}_1\boldsymbol{U}_2^{-1} = \boldsymbol{D}_1$, we have $\boldsymbol{U}_1\boldsymbol{U}_2^{-1} = \boldsymbol{I}$. It then implies that $\boldsymbol{U}_1 = (\boldsymbol{U}_2^{-1})^{-1} = \boldsymbol{U}_2$. ■

**7.** Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices, it implies that $\boldsymbol{A}^T = \boldsymbol{A}$ and $\boldsymbol{B}^T = \boldsymbol{B}$.

(a) We have

$$(\boldsymbol{A}^2)^T = (\boldsymbol{A}\boldsymbol{A})^T = (\boldsymbol{A}^T\boldsymbol{A}^T) = \boldsymbol{A}\boldsymbol{A} = \boldsymbol{A}^2.$$

Therefore, $\boldsymbol{A}^2$ is symmetric, and so is $\boldsymbol{B}^2$. Since

$$(\boldsymbol{A}^2 - \boldsymbol{B}^2)^T = (\boldsymbol{A}^2)^T - (\boldsymbol{B}^2)^T = \boldsymbol{A}^2 - \boldsymbol{B}^2$$

$\boldsymbol{A}^2 - \boldsymbol{B}^2$ is also symmetric.

(b) The product $(\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}-\boldsymbol{B})$ is not always symmetric. A counterexample is given as follows. Consider two symmetric matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and we can have

$$(\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}-\boldsymbol{B}) = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

which is not a symmetric matrix.

(c) Since $(\boldsymbol{ABA})^T = \boldsymbol{A}^T\boldsymbol{B}^T\boldsymbol{A}^T = \boldsymbol{ABA}$, $\boldsymbol{ABA}$ is symmetric.

(d) The product $\boldsymbol{ABAB}$ is not always symmetric. A counterexample is given as follows. Consider two symmetric matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and we can have

$$\boldsymbol{ABAB} = \begin{bmatrix} 19 & 4 & 19 \\ 7 & 2 & 7 \\ 18 & 3 & 18 \end{bmatrix}$$

which is not a symmetric matrix.

**8.** (a) First do row exchange as

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{\boldsymbol{P}_{13}} \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \boldsymbol{PA}$$

and then perform elimination as

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{\boldsymbol{E}_{21}} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -5 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{\boldsymbol{E}_{31}} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -5 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{\boldsymbol{E}_{32}} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -5 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \boldsymbol{U}.$$

Then we have

$$\boldsymbol{E}_{32}\boldsymbol{E}_{31}\boldsymbol{E}_{21}(\boldsymbol{PA}) = \boldsymbol{U}$$

where

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ \boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

Multiplying $\boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\boldsymbol{E}_3^{-1}$ to both sides, we can have

$$\boldsymbol{PA} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\boldsymbol{E}_3^{-1}\boldsymbol{U} = \boldsymbol{LU}$$

where

$$\boldsymbol{L} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\boldsymbol{E}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix}.$$

The factorization $\boldsymbol{PA} = \boldsymbol{LU}$ is hence given by

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -5 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

In order to factor $\boldsymbol{A}$ into $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$, we first perform elimination as

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{\boldsymbol{E}_{21}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{\boldsymbol{E}_{31}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

and then do row exchange as

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix} \xrightarrow{\boldsymbol{P}_{23}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{U_1}.$$

Therefore,

$$\boldsymbol{U_1} = \boldsymbol{P}_1\boldsymbol{E}_{31}\boldsymbol{E}_{21}\boldsymbol{A}$$

where

$$\boldsymbol{P}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ \boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \boldsymbol{E}_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Multiplying $\boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{31}^{-1}\boldsymbol{P}_1^{-1}$ from the left to both sides, we can have

$$\boldsymbol{A} = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{31}^{-1}\boldsymbol{P}_1^{-1}\boldsymbol{U}_1$$

where $\boldsymbol{P}_1^{-1} = \boldsymbol{P}_1$ and

$$\boldsymbol{L}_1 = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The factorization $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$ is hence given by

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Do row exchange and elimination as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 4 & 5 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \overset{\boldsymbol{P}_{13}}{\Longrightarrow} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 5 \end{bmatrix} \overset{\boldsymbol{E}_{32}}{\Longrightarrow} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} = \boldsymbol{U} = \boldsymbol{E}_{32}\boldsymbol{P}\boldsymbol{A}.$$

We can have

$$\boldsymbol{P}\boldsymbol{A} = \boldsymbol{E}_{32}^{-1}\boldsymbol{U} = \boldsymbol{L}\boldsymbol{U}$$

where

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{L} = \boldsymbol{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

The factorization $\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$ is hence given by

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 & 5 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}.$$

In order to factor $\boldsymbol{A}$ into $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$, we perform elimination first as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 4 & 5 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \overset{\boldsymbol{E}_{21}}{\Longrightarrow} \begin{bmatrix} 0 & 4 & 5 \\ 0 & 0 & \frac{3}{4} \\ 2 & 1 & 1 \end{bmatrix} \overset{\boldsymbol{P}'}{\Longrightarrow} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & \frac{3}{4} \end{bmatrix} = \boldsymbol{U_1} = \boldsymbol{P}'\boldsymbol{E_{21}}\boldsymbol{A}.$$

Then we can obtain

$$\boldsymbol{A} = \boldsymbol{E}_{21}^{-1}\boldsymbol{P}'^{-1}\boldsymbol{U}_1$$

where

$$\boldsymbol{P}' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since

$$\boldsymbol{P}_1 = \boldsymbol{P}'^{-1} = \boldsymbol{P}'^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{L}_1 = \boldsymbol{E}'^{-1}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the factorization $\boldsymbol{A} = \boldsymbol{L}_1\boldsymbol{P}_1\boldsymbol{U}_1$ is hence given by

$$\begin{bmatrix} 0 & 4 & 5 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & \frac{3}{4} \end{bmatrix}.$$